Enumeration of Shi regions with a fixed separating wall

Ricardo Mamede
joint work with Alessandro Conflitti and Eleni Tzanaki

The 5th Combinatorics Day, Covilhã

17-4-2015

The extended Shi Arrangement

- $\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}$ standard basis of $\mathbb{R}^{n+1}$
- $\langle \,|\, \rangle$ standard inner product
- $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n$, is a basis of

$$V = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + x_2 + \cdots + x_{n+1} = 0\}.$$

- $\alpha_{ij} = \alpha_i + \cdots + \alpha_j = \varepsilon_i - \varepsilon_{j+1}$, with $\alpha_{ii} = \alpha_i$, and $\theta = \alpha_{1,n}$
- The elements of $\Delta = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ are called roots
  A root α is positive ($\alpha > 0$) if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$.

The extended Shi Arrangement

- For each $\alpha \in \Delta^+$ we define its reflecting hyperplane

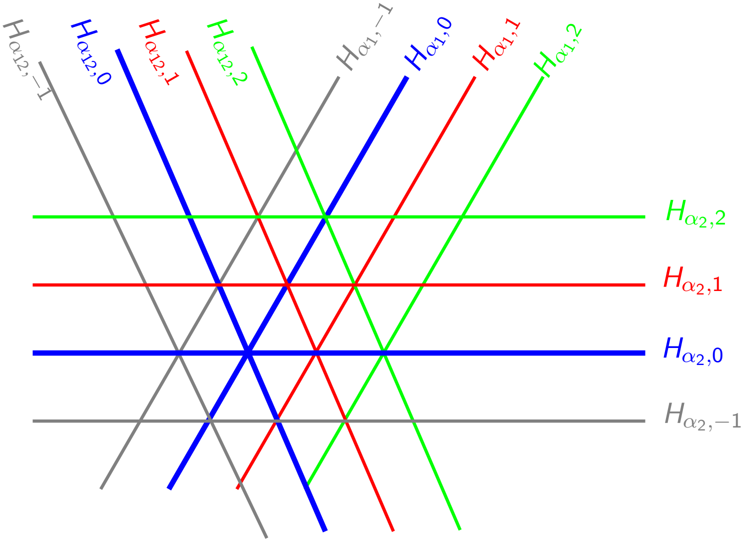
$$H_{\alpha,0} = \{v \in V : \langle v|\alpha\rangle = 0\},$$

and for $k \in \mathbb{Z}$, the $H_{\alpha,0}$'s translate

$$H_{\alpha,k} = \{v \in V : \langle v|\alpha\rangle = k\}.$$

- The extended Shi arrangement, here called the m-Shi arrangement, is

$$\mathcal{H}_m = \{H_{\alpha,k} : \alpha \in \Delta^+, -m < k \leq m\}.$$



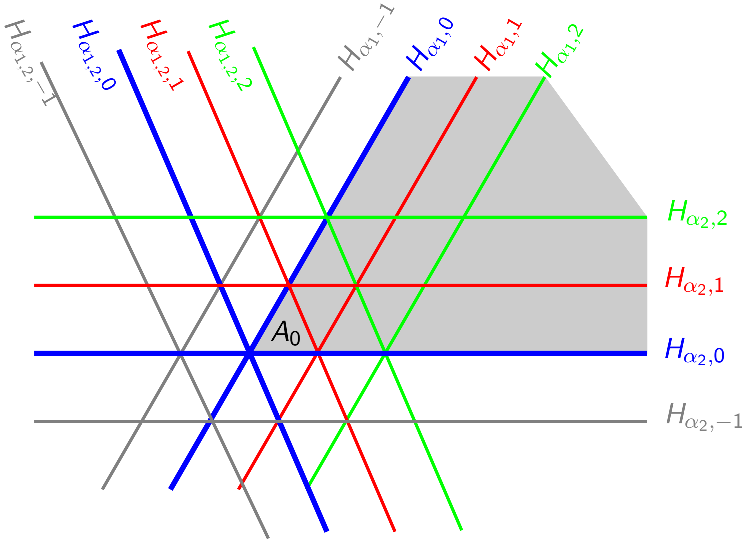
The 2-Shi arrangement $\mathcal{H}_2 = \{H_{\alpha,k} : \alpha \in \Delta^+, -2 < k \leq 2\}$.

- A region is a connected component of $V \setminus \bigcup_{H \in \mathcal{H}_m} H$
- For $\alpha \in \Delta^+$ and $k \in \mathbb{Z}$,

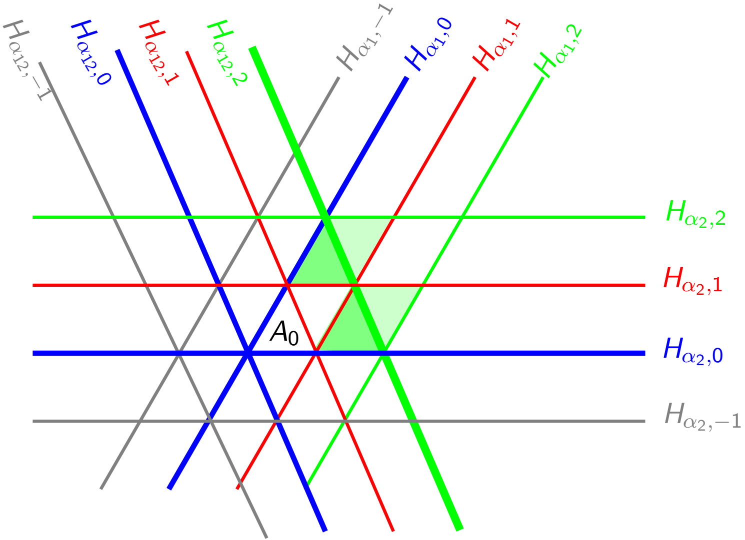
$$H^+_{\alpha,k} = \{v \in V : \langle v|\alpha\rangle \geq k\}.$$

- The dominant chamber of V is $V \cap \bigcap_{i=1}^{n} H^+_{\alpha_i,0}$.

$\mathcal{H}_2$; the dominant chamber is the part with a gray background.



- A dominant region is a region contained in the dominant chamber.
- A wall of a region R is an hyperplane which contains a facet of R.
- A separating wall for a region R is a wall of R which separates R from the region $A_0 := H^-_{\theta,1} \cap \bigcap_{i=1}^n H^+_{\alpha_i,0}$.



$H_{\alpha_{12},-1}$
$H_{\alpha_{12},0}$
$H_{\alpha_{12},1}$
$H_{\alpha_{12},2}$
$H_{\alpha_1,-1}$
$H_{\alpha_1,0}$
$H_{\alpha_1,1}$
$H_{\alpha_1,2}$
$H_{\alpha_2,2}$
$H_{\alpha_2,1}$
$H_{\alpha_2,0}$
$H_{\alpha_2,-1}$
A_0

Each connected component of

$$V \setminus \bigcup_{\substack{\alpha \in \Delta^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$$

ia called an alcove.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts freely and transitively on the set of alcoves. We identify each alcove A with the unique $w \in \widehat{\mathfrak{S}}_n$ such that $A = w^{-1}A_0$.

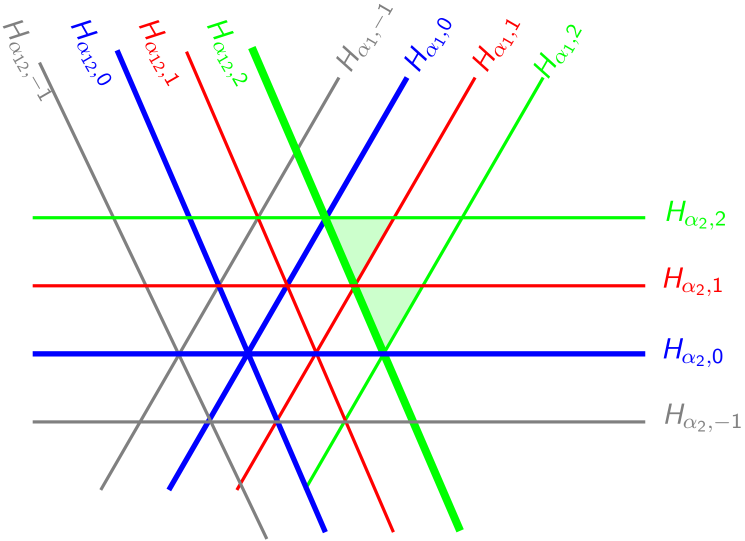
Each simple generator s_i of $\widehat{\mathfrak{S}}_n$ acts by reflection over the hyperplane $H_{\alpha_i,0}$, and s_0 acts as reflection over the hyperplane $H_{\theta,1}$.

Enumerative results - Motivation

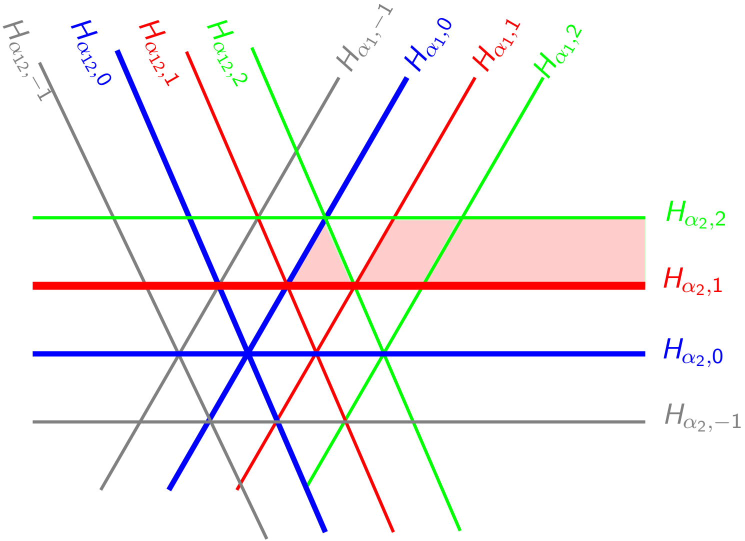
- The number of dominant regions in the m-Shi arrangement is $\frac{1}{mn+1}\binom{(m+1)n}{n}$.
- The number of dominant regions having k separating walls of type $H_{\alpha_{ij},m}$ is equal to the kth m-Narayana number

$$\frac{1}{mn+1}\binom{n-1}{n-k-1}\binom{mn+1}{n-k}.$$

Problem: Fix $\alpha_{ij} \in \Delta^+$ and an integer $0 < k \leq m$. Find the number of dominant regions having separating wall $H_{\alpha_{ij},k}$.



Problem: Fix $\alpha_{ij} \in \Delta^+$ and an integer $0 < k \leq m$. Find the number of dominant regions having separating wall $H_{\alpha_{ij},k}$.



Every alcove A can be written as $w^{-1}A_0$ for a unique $w \in \widehat{\mathfrak{S}}_n$, and additionally, for each $\alpha \in \Delta^+$, there is a unique integer k_α such that

$$k_\alpha < \langle \alpha, x \rangle < k_\alpha + 1,$$

for all $x \in A$.

Every alcove A can be written as $w^{-1}A_0$ for a unique $w \in \widehat{\mathfrak{S}}_n$, and additionally, for each $\alpha \in \Delta^+$, there is a unique integer k_α such that

$$k_\alpha < \langle \alpha, x \rangle < k_\alpha + 1,$$

for all $x \in A$.

Lemma (Shi,1987)

Let $\{k_{\alpha_{ij}}\}_{1 \leq i \leq j \leq n-1}$ be a set of $\binom{n}{2}$ integers. There exists $w \in \widehat{\mathfrak{S}}_n$ such that

$$k_{\alpha_{ij}} < \langle \alpha_{ij}, x \rangle < k_{\alpha_{ij}} + 1,$$

for all $x \in w^{-1}A_0$ if and only if

$$k_{\alpha_{it}} + k_{\alpha_{t+1,j}} \leq k_{\alpha_{ij}} \leq k_{\alpha_{it}} + k_{\alpha_{t+1,j}} + 1,$$

for all t such that $i \leq t < j$.

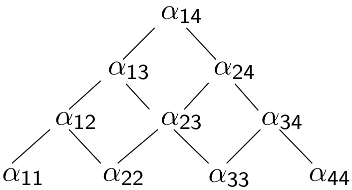
The root poset and m-Shi tableaux

- Root order on Δ^+: $\alpha \leq \beta$ if $\beta - \alpha$ is in the integer span of Π.

$$\begin{tikzcd}[column sep=small]
 & & & \alpha_{14} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & \alpha_{13} \arrow[dl, dash] \arrow[dr, dash] & & \alpha_{24} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \alpha_{12} \arrow[dl, dash] \arrow[dr, dash] & & \alpha_{23} \arrow[dl, dash] \arrow[dr, dash] & & \alpha_{34} \arrow[dl, dash] \arrow[dr, dash] & \\
\alpha_{11} & & \alpha_{22} & & \alpha_{33} & & \alpha_{44}
\end{tikzcd}$$

The root poset and m-Shi tableaux

- Root order on Δ^+: $\alpha \leq \beta$ if $\beta - \alpha$ is in the integer span of Π.



α_{14}	α_{13}	α_{12}	α_{11}
α_{24}	α_{23}	α_{22}	
α_{34}	α_{33}		
α_{44}			

- Arrange the roots α_{ij}, $1 \leq i < j \leq n$ in a n-staircase diagram such that α_{ij} is in box $(i, n-i+1)$.

The root poset and m-Shi tableaux

- To each root α_{ij} we associate an integer $0 \leq k_{ij} \leq m$

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

- The filling of this n-staircase diagram is called an **m-Shi tableau** if the following two conditions hold:

$$k_{ij} = \begin{cases} k_{i\ell} + k_{\ell+1,j} \text{ or } k_{i\ell} + k_{\ell+1,j} + 1, & \text{if } k_{i\ell} + k_{\ell+1,j} \leq m-1 \\ & \quad \text{and } i \leq \ell < j \\ m, & \text{otherwise.} \end{cases}$$

Example of a 4-Shi tableau

3	2	1	1
2	1	0	
2	1		
1			

- For each $k_{ij} \leq m - 1$ we check if the sum of the values of the endpoints of each hook on k_{ij} of length $j - i + 2$ sum up to k_{ij} or $k_{ij} - 1$.
- For each $k_{ij} = m$ we check if the sum of the values of the endpoints of each hook on k_{ij} of length $j - i + 2$ sum up to a value $\geq m$.

Example of a 4-Shi tableau

Checking if the entries of the first row are valid:

3	2	1	1
2	1	0	
2	1		
1			

3	2	1	1
2	1	0	
2	1		
1			

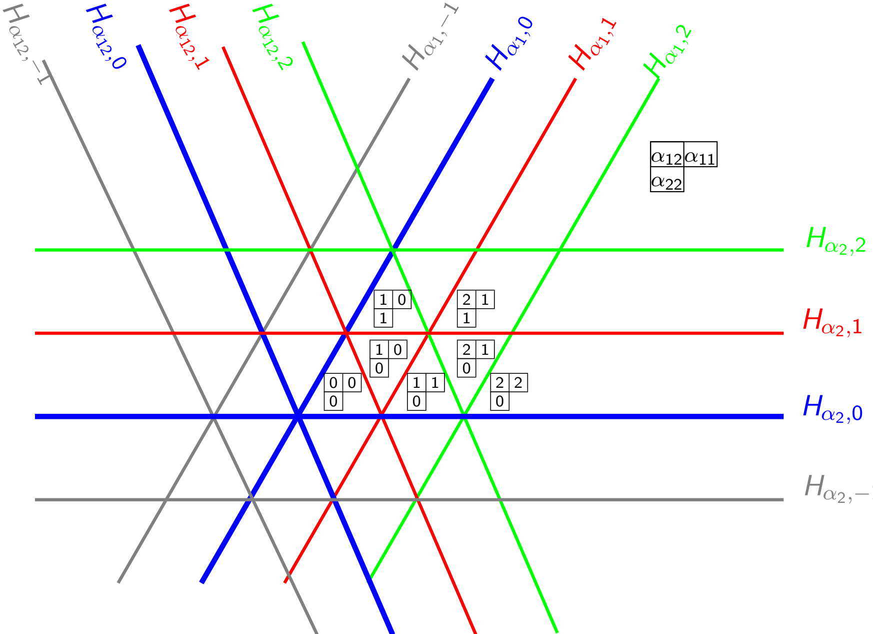
3	2	1	1
2	1	0	
2	1		
1			

Theorem (Athanasiadis)

The m-Shi tableaux are in bijection with the dominant regions in the m-Shi arrangement.

The bijection:

- Given a dominant region R in the m-Shi arrangement, k_{ij} is the number of integer translates of $H_{\alpha_{ij},0}$ that separates R from the origin.
- Given an m-Shi tableau, the corresponding region R consists of those points x such that $\langle \alpha_{ij}, x \rangle \geq k_{ij}$, for all $1 \leq i \leq j \leq n$.



$H_{\alpha_{12},-1}$
$H_{\alpha_{12},0}$
$H_{\alpha_{12},1}$
$H_{\alpha_{12},2}$
$H_{\alpha_1,-1}$
$H_{\alpha_1,0}$
$H_{\alpha_1,1}$
$H_{\alpha_1,2}$
α_{12} α_{11}
α_{22}
$H_{\alpha_2,2}$
$H_{\alpha_2,1}$
$H_{\alpha_2,0}$
1 0 / 1
2 1 / 1
1 0 / 0
2 1 / 0
0 0 / 0
1 1 / 0
2 2 / 0

m-Shi tableaux and separating walls

Theorem

Let T_R be the m-Shi tableau for the region R. The hyperplane $H_{\alpha_{ij},m}$ is a separating wall for the region R if and only if $k_{ij} = m$ and $k_{i\ell} + k_{\ell+1,j} = m - 1$ for all $i \leq \ell < j$.

Problem:

- Count the number of regions having $H_{\alpha_{ij},m}$ as separating wall.
- Count the number of m-Shi tableux with $k_{ij} = m$ and $k_{i\ell} + k_{\ell+1,j} = m - 1$ for all $i \leq \ell < j$.

Base Case

Theorem (Fishel,Tzanaki,Vazirani)
The number of regions in the m-Shi arrangement $\mathcal{H}_m$ with separating wall $H_{\theta,m}$ is equal to m^{n-1}.

- Let $w = w_1 \cdots w_{n-1}$ be a word over the alphabet $\{0, 1, \ldots, m-1\}$, and let $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_{n-1}$ be its rearrangement in increasing order.
- Define the position vector $\mathrm{Pos}(\widetilde{w}) = (\mathrm{Pos}(\widetilde{w}_1), \ldots, \mathrm{Pos}(\widetilde{w}_{n-1})$ as follows: if $j = 1$ or $\widetilde{w}_j > \widetilde{w}_{j-1}$

$$\mathrm{Pos}(\widetilde{w}_j) = \min\{1 \leq l \leq n-1 \text{ such that } w_l = \widetilde{w}_j\},$$

whereas if $j > 1$ and $\widetilde{w}_j = \widetilde{w}_{j-1}$

$$\mathrm{Pos}(\widetilde{w}_j) = \min\{\mathrm{Pos}(\widetilde{w}_{j-1}) + 1 \leq l \leq n-1 \text{ such that } w_l = \widetilde{w}_j\}.$$

- Let $w = w_1 \cdots w_{n-1}$ be a word over the alphabet $\{0, 1, \ldots, m-1\}$, and let $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_{n-1}$ be its rearrangement in increasing order.
- Define the position vector $\mathrm{Pos}(\widetilde{w}) = (\mathrm{Pos}(\widetilde{w}_1), \ldots, \mathrm{Pos}(\widetilde{w}_{n-1})$ as follows: if $j = 1$ or $\widetilde{w}_j > \widetilde{w}_{j-1}$

$$\mathrm{Pos}(\widetilde{w}_j) = \min\{1 \leq l \leq n-1 \text{ such that } w_l = \widetilde{w}_j\},$$

  whereas if $j > 1$ and $\widetilde{w}_j = \widetilde{w}_{j-1}$

$$\mathrm{Pos}(\widetilde{w}_j) = \min\{\mathrm{Pos}(\widetilde{w}_{j-1})+1 \leq l \leq n-1 \text{ such that } w_l = \widetilde{w}_j\}.$$

For example, if $w = 61513$ then $\widetilde{w} = 11356$ and $\mathrm{Pos}(\widetilde{w}) = (2, 4, 5, 3, 1)$.

- We set $k_{1,n} = m$, and for all $j = 1, \ldots, n-1$ we set

$$k_{1,j} = \widetilde{w}_j$$

and

$$k_{j+1,n-1} = m - 1 - k_{1,j} = m - 1 - \widetilde{w}_j.$$

- We set $k_{1,n} = m$, and for all $j = 1, \dots, n-1$ we set

$$k_{1,j} = \widetilde{w}_j$$

and

$$k_{j+1,n-1} = m - 1 - k_{1,j} = m - 1 - \widetilde{w}_j.$$

- For all $t = 2, \dots, n-1$ and all $j = t, \dots, n-1$ we set

$$k_{t,j} = \begin{cases} k_{1,j} - k_{1,t-1} & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) < \operatorname{Pos}(\widetilde{w}_j) \\ k_{1,j} - k_{1,t-1} - 1 & \text{if } \operatorname{Pos}(\widetilde{w}_{t-1}) > \operatorname{Pos}(\widetilde{w}_j); \end{cases}$$

Ex: $w = 61513 \in \{0, 1, 2, 3, 4, 5, 6\}^*$, with $\widetilde{w} = 11356$ and $\mathrm{Pos}(\widetilde{w}) = (2, 4, 5, 3, 1)$; the corresponding 7-Shi tableau is

7					

Ex: $w = 61513 \in \{0,1,2,3,4,5,6\}^*$, with $\widetilde{w} = 11356$ and $\mathrm{Pos}(\widetilde{w}) = (2,4,5,3,1)$; the corresponding 7-Shi tableau is

7					

$\rightarrow$

7	6	5	3	1	1
5					
5					
3					
1					
0					

Ex: $w = 61513 \in \{0,1,2,3,4,5,6\}^*$, with $\widetilde{w} = 11356$ and $\mathrm{Pos}(\widetilde{w}) = (2,4,5,3,1)$; the corresponding 7-Shi tableau is

$$
\begin{array}{cccccc}
7 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \\
\square & \square & \square & \square & & \\
\square & \square & \square & & & \\
\square & \square & & & & \\
\square & & & & &
\end{array}
\;\to\;
\begin{array}{cccccc}
7 & 6 & 5 & 3 & 1 & 1 \\
5 & \square & \square & \square & \square & \\
5 & \square & \square & \square & & \\
3 & \square & \square & & & \\
1 & \square & & & & \\
0 & & & & &
\end{array}
\;\to\;
\begin{array}{cccccc}
7 & 6 & 5 & 3 & 1 & 1 \\
5 & 4 & 4 & 2 & 0 & \\
5 & \square & \square & \square & & \\
3 & \square & \square & & & \\
1 & \square & & & & \\
0 & & & & &
\end{array}
$$

Ex: $w = 61513 \in \{0, 1, 2, 3, 4, 5, 6\}^*$, with $\widetilde{w} = 11356$ and $\mathrm{Pos}(\widetilde{w}) = (2, 4, 5, 3, 1)$; the corresponding 7-Shi tableau is

7					

$\rightarrow$

7	6	5	3	1	1
5					
5					
3					
1					
0					

$\rightarrow$

7	6	5	3	1	1
5	4	4	2	0	
5					
3					
1					
0					

7	6	5	3	1	1
5	4	4	2	0	
5	4	3	2		
3	2	1			
1	0				
0					

Consider the tableaux $\begin{array}{cccccc} 7 & 6 & 5 & 3 & 1 & 1 \\ 5 & 4 & 4 & 2 & 0 & \\ 5 & 4 & 3 & 2 & & \\ 3 & 2 & 1 & & & \\ 1 & 0 & & & & \\ 0 & & & & & \end{array}$; then $\widetilde{w} = 11356$ and the inversion table is $I = \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & \\ 1 & 1 & & \\ 1 & & & \end{array}$.

Consider the tableaux $\begin{array}{|c|c|c|c|c|c|}\hline 7&6&5&3&1&1\\\hline 5&4&4&2&0\\\cline{1-5} 5&4&3&2\\\cline{1-4} 3&2&1\\\cline{1-3} 1&0\\\cline{1-2} 0\\\cline{1-1}\end{array}$; then $\widetilde{w} = 11356$ and the inversion table is $I = \begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 1&1&0\\\cline{1-3} 1&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$.

- Row 1: $\mathrm{Pos}(6) < \mathrm{Pos}(1)$ and $\mathrm{Pos}(5), \mathrm{Pos}(3), \mathrm{Pos}(1) > \mathrm{Pos}(1)$. Thus

$$w = 61\{1, 3, 5\}$$

Consider the tableaux $\begin{array}{cccccc} 7 & 6 & 5 & 3 & 1 & 1 \\ 5 & 4 & 4 & 2 & 0 & \\ 5 & 4 & 3 & 2 & & \\ 3 & 2 & 1 & & & \\ 1 & 0 & & & & \\ 0 & & & & & \end{array}$; then $\widetilde{w} = 11356$ and the inversion table is $I = \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & \\ 1 & 1 & & \\ 1 & & & \end{array}$.

- Row 1: $\text{Pos}(6) < \text{Pos}(1)$ and $\text{Pos}(5), \text{Pos}(3), \text{Pos}(1) > \text{Pos}(1)$. Thus

$$w = 61\{1, 3, 5\}$$

- Row 2: $\text{Pos}(6), \text{Pos}(5) < \text{Pos}(1)$ and $\text{Pos}(3) > \text{Pos}(1)$. Thus

$$w = 61513.$$

Towards the general case

- Let $h^n_{\alpha,k}$ be the set of dominant regions in the m-Shi arrangement having $H_{\alpha,k}$ as a separating wall.
- Given a fundamental region R, let

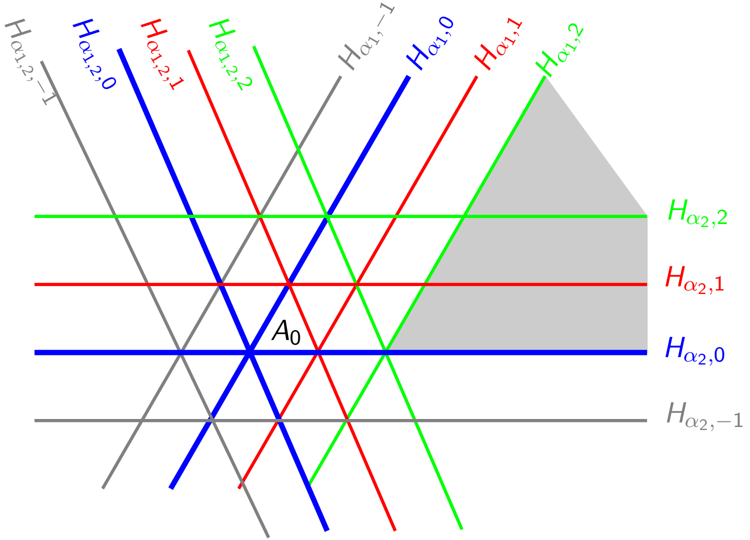
$$r(R) = \#\{(j,k) : R \text{ and } A_0 \text{ are separated by } H_{\alpha_{1j}k}, 1 \leq k \leq m\}$$

$$c(R) = \#\{(i,k) : R \text{ and } A_0 \text{ are separated by } H_{\alpha_{i,n-1}k}, 1 \leq k \leq m\}$$

- The generating function is

$$f^n_{\alpha_{ij}m}(p,q) = \sum_{R \in h^n_{\alpha_{ij}k}} p^{c(R)} q^{r(R)}.$$

$$f^3_{\alpha_1,2}(p,q) = p^4q^2 + p^4q^3 + p^4q^4.$$



- C. A. Athanasiadis. On a refinement of the generalized Catalan numbers for Weyl groups. Trans. Amer. Math. Soc., 357(1):179–196 (electronic), 2005
- S. Fishel, E. Tzanaki and M. Vazirani, Counting Shi regions with a fixed separating wall, DMTCS proc. AO, 2011, 351–362.
- J. Y. Shi. The Kazhdan-Lusztig cells in certain affine Weyl groups, volume 1179 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.
- R. P. Stanley. Hyperplane arrangements, parking functions and tree inversions. In Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), volume 161 of Progr. Math., pages 359–375. Birkhauser Boston, Boston, MA, 1998
- A. Conflitti, R. Mamede, E. Tzanaki,Dominant Shi regions with a fixed separating wall: bijective enumeration, The Australasian Journal of Combinatorics, Volume 60 (2014), pp 198-207.